2024 Stuart Sidney Undergraduate Math Competition

August 13, 2024

1. Prove that $\binom{2024}{1000}$ is odd.

   Proof. We have $\binom{2024}{1000} = \binom{2024}{1024}$. More generally we can prove that if $2^n \le m \le 2^{n+1} - 1$, then $\binom{m}{2^n}$ is odd. This is an easy application of Lucas' Theorem on binomial coefficients modulo a prime, but we can give a direct elementary proof.

   Let $m = 2^n + r$ where $0 \le r \le 2^n - 1$. We have $\binom{m}{2^n} = \frac{(r+2^n)(r+2^n-1)\cdots(r+1)}{(2^n)!}$. Denote this expression by $f(r)$. We have $f(0) = 1$ and $f(r+1) = \frac{r+2^n+1}{r+1} f(r)$.

   Claim: If $r < 2^n$, then $\frac{r+2^n}{r} = \frac{a}{b}$ for some *odd* integers a and b.

   Assuming the claim, our desired statement follows by induction: $f(r+1)b = f(r)a$ with a, b odd implies that $f(r)$ and $f(r+1)$ have the same parity.

   For the claim, write $r = 2^k s$ where s is odd. We have $k < n$ since $r < 2^n$. Then $\frac{r+2^n}{r} = \frac{s+2^{n-k}}{s}$. Clearly s and $s + 2^{n-k}$ are odd. □

2. Let $f : [0, 2\pi] \to \mathbb{R}$ be a continuous function which is differentiable on $(0, 2\pi)$. Show that there exists some $\xi \in (0, 2\pi)$ such that $(\cos\xi, \sin\xi)$ is orthogonal to $(f(\xi), f'(\xi) - 2024)$.

   Proof. We need to find $\xi \in (0, 2\pi)$ such that

   $$f(\xi)\cos\xi + (f'(\xi) - 2024)\sin\xi = f(\xi)\cos\xi + f'(\xi)\sin\xi - 2024\sin\xi = F'(\xi) = 0.$$

   where

   $$F(x) = f(x)\sin x + 2024\cos x.$$

   Note that

   $$F(0) = 2024 = F(2\pi).$$

   So, by Rolle's Theorem there exists ξ such that $F'(\xi) = 0$. □

3. Determine which of the numbers 50^{50} and 49^{51} is larger (if any).

Proof. 49^{51} is the largest number. First note that

$$\frac{49^{51}}{50^{50}} = 49\left(\frac{49}{50}\right)^{50} = 49\left(1-\frac{1}{50}\right)^{50} = 49\left(\left(1-\frac{1}{50}\right)^{25}\right)^{2}$$

Recall Bernoulli's inequality:

$$(1+x)^n \geq 1+nx \text{ for } x > -1, n \in \mathbb{N}.$$

Hence,

$$\left(1-\frac{1}{50}\right)^{25} \geq 1-\frac{25}{50} = \frac{1}{2}.$$

Therefore,

$$\frac{49^{51}}{50^{50}} \geq \frac{49}{4} > 12.$$

□

4. Let A, B be 2×2 matrices with real entries. Prove that there exist real numbers a, b, c, d, e at least one of which is nonzero such that

$$A(aB + bI_2) + B(cA + dI_2) = eI_2$$

where $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Proof. The set of 2×2 matrices with real entries is a 4-dimensional vector space with basis given by the 4 matrices that have 1 in some corner and 0 elsewhere. Then the 5 matrices AB, A, BA, B, I_2 must be linearly dependent. The conclusion can be expressed as $aAB + bA + cBA + dB - eI_2 = 0_2$ □

5. Find the limit

$$\lim_{n\to\infty} \int_n^{2n} \frac{x^2 - 2x\log x - 1}{x(x-1)^2}\,\mathrm{d}x.$$

Proof. We set

$$A_n = \int_n^{2n} \frac{x^2-1}{x(x-1)^2}\,\mathrm{d}x$$

and

$$B_n = 2\int_n^{2n} \frac{\log x}{(x-1)^2}\,\mathrm{d}x.$$

First, using partial fractions

$$A_n = \int_n^{2n} \left(\frac{2}{x-1} - \frac{1}{x} \right) \mathrm{d}x = 2\log\left(\frac{2n-1}{n-1}\right) - \log\frac{2n}{n} = 2\log\left(\frac{2n-1}{n-1}\right) - \log 2.$$

Thus

$$\lim_{n\to\infty} A_n = 2\log 2 - \log 2 = \log 2.$$

For B_n:

$$0 \le B_n \le 2\log(2n) \int_n^{2n} \frac{1}{(x-1)^2} \mathrm{d}x = 2\log(2n)\left(\frac{1}{n-1} - \frac{1}{2n-1}\right) = 2\frac{\log 2n}{n-1}\frac{n}{2n-1}.$$

It is easy to check using L'Hospital's rule that

$$\lim_{n\to\infty} \frac{\log 2n}{n-1}\frac{n}{2n-1} = 0.$$

Hence, an application of the sqeeuze theorem for sequences gives us that $B_n \to 0$. Thus,

$$\lim_{n\to\infty} \int_n^{2n} \frac{x^2 - 2x\log x - 1}{x(x-1)^2} \mathrm{d}x = \log 2.$$

□

6. Let $p(x) = x^3 + a_2x^2 + a_1x + a_0$ with a_0, a_1, a_2 integers. Prove that if $p(2023)$, $p(2024)$, $p(2025)$ are multiples of 3, then a_0 and a_2 are multiples of 3.

Proof. Since 2025 is a multiple of 3, we deduce that $a_0 = p(2025) - 2025(2025a_2 + a_1)$ is also a multiple of 3. Let

$$q(x) = x^2 + a_2x + a_1.$$

Since 2023 and 2024 are not multiples of 3, it follows from $p(x) - a_0 = x \cdot q(x)$ that $q(2023)$ and $q(2024)$ are multiples of 3. Then $q(2024) - q(2023) = 4047 + a_2$ is divisible by 3. Since $3 \mid 4047$, we obtain the conclusion. □

7. Evaluate

$$\int_0^1 \frac{\mathrm{d}x}{\sqrt{x} + \sqrt[3]{x}}.$$

Proof. We will use the change of variables $x = y^6$. Therefore (using also partial fraction decomposition)

$$\int \frac{\mathrm{d}x}{\sqrt{x} + \sqrt[3]{x}} \overset{x=y^6}{=} \int \frac{6y^5}{y^3 + y^2} \mathrm{d}y = \int \frac{6y^3}{y+1} \mathrm{d}y = 6\int \left(y^2 - y - \frac{1}{y+1} + 1 \right) \mathrm{d}y.$$

Therefore,

$$\int_0^1 \frac{\mathrm{d}x}{\sqrt{x}+\sqrt[3]{x}} = 6\left(\frac{y^3}{3}-\frac{y^2}{2}-\log(y+1)+y\right)\Big|_0^1$$
$$= \left(2\sqrt{x}-3\sqrt[3]{x}-6\log(\sqrt[6]{x}+1)+6\sqrt[6]{x}\right)\Big|_0^1 = 5-6\log 2.$$

□

8. Consider a linear system of equations

$$\begin{cases} ax+by=c \\ dx+ey=f \end{cases}$$

where a,b,c,d,e,f are rational numbers and x,y are unknowns. Prove that if the system has a real solution (x,y), then it also has a solution where x,y are rational.

Proof. If $a=b=d=e=0$ then the system having a real solution also forces $c=f=0$, thus any pair of rational numbers (x,y) is a solution.

Assume that a,b,d,e are not all 0. Up to swapping the equations, and swapping x,y we may assume that $a\neq 0$. The system is equivalent then to

$$\begin{cases} x+\frac{b}{a}y=\frac{c}{a} \\ (e-\frac{bd}{a})y=f-\frac{bc}{a} \end{cases}$$

Either $e-\frac{bd}{a}\neq 0$ and then y is uniquely determined and rational and so is x, or $e-\frac{bd}{a}=0$. In this case we must also have $f=\frac{bc}{a}$ for the system to have solutions. Any rational choice for y leads to a solution where x is also rational. □

9. Let $f:\mathbb{R}\to\mathbb{R}$ be a function such that

$$f(f(x))=-x \text{ for all } x\in\mathbb{R}.$$

Show that f is not continuous in $\mathbb{R}$.

Proof. First note that f is one to one and onto. Indeed, if $f(x)=f(y)$ then

$$-x=f(f(x))=f(f(y))=-y,$$

so $x=y$. Therefore f is one to one. Now let $y\in\mathbb{R}$. Then

$$f(f(-y))=y.$$

So f is onto.

Since f is a continuous bijection, f is either strictly increasing or strictly decreasing. You can verify this easily using the Intermediate Value Theorem. However, in both cases $f\circ f$ would be strictly increasing. But this is a contradiction because $f\circ f(x)=-x$ is strictly decreasing. □

10. You have two standard chess knights of the same color. In how many distinct ways can you place them in different squares on an empty standard numbered 8×8 chess board so that they don't attack each other? Recall that the knight moves two squares in vertical or horizontal direction, and one square either way in the perpendicular direction. Since the knights have the same color, two configurations that differ only by swapping the knights should only be counted once. Since the board is numbered, rotating a configuration of knights will produce a different configuration. You do not need to simplify your final answer.

Proof. There are $\binom{64}{2} = 2016$ distinct configurations of two knights not sitting on the same square. They may or may not attack each other. Let's count how many configurations do attack each other. For this first we pretend that the knights are of different color, then count attacking configurations, and then divide by two. The number of squares that a knight attacks is as follows:

- Any of the 4 corner squares: Only attacks two squares.
- Any of the 8 side squares adjacent to the corners: Only attacks 3 squares.
- The other 16 side squares attack 4 squares.
- Any of the 4 squares diagonally adjacent to corners: attack 4 squares.
- Any of the other 16 squares adjacent to a side square attacks 6 squares.
- Any of the other 16 "central" squares attacks 8 squares.

Altogether we get $4\cdot2+8\cdot3+16\cdot4+4\cdot4+16\cdot6+16\cdot8 = 4\cdot2+8\cdot3+20\cdot4+16\cdot6+16\cdot8 = 336$ "ordered" attacking configurations, so 168 unordered attacking configurations. The final answer is $2016 - 168 = 1848$. □