Stuart Sidney Math Competition **Name (Print):** ____________________
Spring 2024
04/02/24
Time Limit: 120 Minutes

This exam contains 3 pages (including this cover page) and 10 problems. Check to see if any pages are missing. Write your name on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use books, notes, or any electronic device on this exam.

Problem	Points	Score
1	5	
2	5	
3	5	
4	5	
5	5	
6	5	
7	5	
8	5	
9	5	
10	5	
Total:	50	

1. (5 points) Prove that $\binom{2024}{1000}$ is odd.

2. (5 points) Let $f : [0, 2\pi] \to \mathbb{R}$ be a continuous function which is differentiable on $(0, 2\pi)$. Show that there exists some $\xi \in (0, 2\pi)$ such that $(\cos\xi, \sin\xi)$ is orthogonal to $(f(\xi), f'(\xi) - 2024)$.

3. (5 points) Determine which of the numbers 50^{50} and 49^{51} is larger (if any).

4. (5 points) Let A, B be 2×2 matrices with real entries. Prove that there exist real numbers a, b, c, d, e at least one of which is nonzero such that

$$A(aB + bI_2) + B(cA + dI_2) = eI_2$$

where $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

5. (5 points) Find the limit

$$\lim_{n\to\infty} \int_n^{2n} \frac{x^2 - 2x\log x - 1}{x(x-1)^2}\, \mathrm{d}x.$$

6. (5 points) Let $p(x) = x^3 + a_2x^2 + a_1x + a_0$ with a_0, a_1, a_2 integers. Prove that if $p(2023)$, $p(2024)$, $p(2025)$ are multiples of 3, then a_0 and a_2 are multiples of 3.

7. (5 points) Evaluate

$$\int_0^1 \frac{\mathrm{d}x}{\sqrt{x} + \sqrt[3]{x}}.$$

8. (5 points) Consider a linear system of equations

$$\begin{cases} ax + by = c \\ dx + ey = f \end{cases}$$

where a, b, c, d, e, f are rational numbers and x, y are unknowns. Prove that if the system has a real solution (x, y), then it also has a solution where x, y are rational.

9. (5 points) Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that

$$f(f(x)) = -x \text{ for all } x \in \mathbb{R}.$$

Show that f is not continuous in $\mathbb{R}$.

10. (5 points) You have two standard chess knights of the same color. In how many distinct ways can you place them in different squares on an empty standard numbered 8×8 chess board so that they don't attack each other? Recall that the knight moves two squares in vertical or horizontal direction, and one square either way in the perpendicular direction. Since the knights have the same color, two configurations that differ only by swapping the knights should only be counted once. Since the board is numbered, rotating a configuration of knights will produce a different configuration. You do not need to simplify your final answer.