2023 Stuart Sidney Undergraduate Math Competition

May 4, 2023

1. Show that $\sin x + \tan x > 2x$ for all $x \in (0, \pi/2)$.

   Solution. Let $f(x) = \sin x + \tan x - 2x$. Then

$$\begin{aligned}
f'(x) &= \cos x + \frac{1}{\cos^2 x} - 2 = \frac{\cos^3 x - 2\cos^2 x + 1}{\cos^2 x} = \frac{\cos^3 x - \cos^2 x + 1 - \cos^2 x}{\cos^2 x} \\
&= \frac{\cos^3 x - \cos^2 x + 1 - \cos^2 x}{\cos^2 x} = \frac{-\cos^2 x(1-\cos x) + (1-\cos x)(1+\cos x)}{\cos^2 x} \\
&= \frac{(1-\cos x)(1+\cos x - \cos^2 x)}{\cos^2 x}.
\end{aligned}$$

   So, since $\cos x < 1$ on $(0, \pi/2)$, we have that $f'(x) > 0$ on $(0, \pi/2)$ and thus f is strictly increasing in $[0, \pi/2)$. Thus, $f(x) > f(0) = 0$ for all $x \in (0, \pi/2)$. □

2. Let $n \in \mathbb{N}$. Evaluate

$$\int_0^{\pi/2} \frac{\sin^n x}{\cos^n x + \sin^n x}\, \mathrm{d}x.$$

   Solution. Let

$$I = \int_0^{\pi/2} \frac{\sin^n x}{\cos^n x + \sin^n x}\, \mathrm{d}x.$$

   After changing variables $x \mapsto x - \pi/2$, and using the trig identities $\sin(\pi/2 - x) = \cos x$ and $\cos(\pi/2 - x) = \sin x$ we get that

$$I = \int_0^{\pi/2} \frac{\cos^n x}{\cos^n x + \sin^n x}\, \mathrm{d}x.$$

   So,

$$2I = \int_0^{\pi/2} \frac{\cos^n x + \sin^n x}{\cos^n x + \sin^n x}\, \mathrm{d}x = \pi/2,$$

   and thus $I = \pi/4$. □

3. Let $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $R_\theta(a,b) = (a\cos\theta - b\sin\theta, a\sin\theta + b\cos\theta)$. This is the counterclockwise rotation by θ degrees around the origin $(0,0)$. Let $M_x(a,b) = (a,-b)$ and $M_y(a,b) = (-a,b)$ be the reflections across the x and y-axes respectively. Let
$$T(a,b) = R_{45}(M_x(R_{90}(M_y(R_{45}(a,b))))).$$
Define $T^2(a,b) = T(T(a,b))$ and similarly define $T^n(a,b)$ for all n. Compute
$$T^{2023}(a,b)$$
in terms of a,b.

Proof. Rotations and reflections are linear transformations from $\mathbb{R}^2$ to $\mathbb{R}^2$ hence so is the composition T. Linear transformations are determined by their values on the standard basis vectors $(1,0)$ and $(0,1)$. By following the composition of geometric transformations, $T(1,0) = (-1,0)$ and $T(0,1) = (0,-1)$. The linear transformation that verifies both is $T(a,b) = (-a,-b)$, the reflection across the origin. Iterating this an odd number of times gives $T^{2023} = T$, hence $T^{2023}(a,b) = T(a,b) = (-a,-b)$. $\square$

4. Let $a_n = \sum_{k=1}^{n} \frac{1}{k}$.
   (a) Show that the limit
   $$\gamma := \lim_{n\to\infty} \left(a_n - \log n\right)$$
   exists in $\mathbb{R}$.
   (b) Find the limit
   $$\lim_{n\to\infty} e^{a_{n+1}} - e^{a_n}.$$

Solution. The proof of (4a) is a direct application of the integral test. For (4b) we argue as follows:
$$\begin{aligned} e^{a_{n+1}} - e^{a_n} &= e^{a_n + \frac{1}{n+1}} - e^{a_n} \\ &= e^{a_n}\left(e^{\frac{1}{n+1}} - 1\right) \\ &= e^{a_n - \log n + \log n}\left(e^{\frac{1}{n+1}} - 1\right) \\ &= e^{a_n - \log n}\, n\left(e^{\frac{1}{n+1}} - 1\right). \end{aligned} \tag{1}$$
Since,
$$n\left(e^{\frac{1}{n+1}} - 1\right) \to 1$$
and by (4a),
$$a_n - \log n \to \gamma,$$
we deduce that
$$\lim_{n\to\infty} e^{a_{n+1}} - e^{a_n} = e^{\gamma}.$$
$\square$

5. Determine which of the digits $0,1,2,3,4,5,6,7,8$ or 9 may occur as the last digit of n^n where n is a positive integer.

   Solution. We prove that the possible last digits are $0,1,3,4,5,6,7,9$. In particular 2 and 8 cannot occur.

   Denote by $L(N)$ the last digit of N. Clearly $L(N) = L(N+10k)$ for every nonnegative integer k and $L(MN) = L(L(M)L(N))$ for every M, N.

   If n is not divisible by 2 or by 5, then $n^4 - 1$ is divisible by 10 so $L(n^4) = 1$. This is an instance of Euler's Theorem that here can be checked directly since the last possible digit of n would be $1,3,7$ or 9. Thus if n ends in $1,3,7,9$, and $n = 4k + r$ with $0 \le r \le 3$, then $L(n^n) = L(n^r)$. Note that if n is coprime to 10, then so are all its powers, thus if $L(n) \in \{1,3,7,9\}$, then $L(n^n) \in \{1,3,7,9\}$. Which of these are possible? We see that the last digit of $1^1, 3^3, 7^7, 9^9$ is $1,7,3,9$.

   What if n is divisible by 2 or by 5? We have for instance $L(10^{10}) = 0$, $L(2^2) = 4$, $L(4^4) = 6$, $L(5^5) = 5$, $L(6^6) = 6$, $L(8^8) = L(16^4) = L(6^4) = 6$. How about 2 and 8?

   For $L(n^n) = 2$ or 8 we would need $n = 2m$ to be even and for $n^n = (n^m)^2$ to be of form $5k+2$ or $5k+3$. But $(n^m)^2$ is a perfect square. The remainders modulo 5 of perfect squares are $0,1,4$. This leaves 2 and 3 out, so 2 or 8 are never the last digit of n^n. $\square$

6. Prove that for all $n \in \mathbb{N}$,

$$\arctan(n+1) - \arctan(n) < \frac{1}{n^2+n}.$$

   Solution. By the Fundamental Theorem of Calculus,

$$\begin{aligned}
\arctan(n+1) - \arctan(n) &= \int_n^{n+1} \frac{1}{1+x^2}\,\mathrm{d}x \\
&< \int_n^{n+1} \frac{1}{x^2}\,\mathrm{d}x \\
&= -\frac{1}{x}\Big|_n^{n+1} = \frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)}.
\end{aligned}$$

$\square$

7. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that $f(0) = 2023$ and $f(2023) = 0$. Show that there exist $a, b \in \mathbb{R}$, $a \neq b$, such that $f'(a)f'(b) = 1$.

   Proof. Let $g(x) = f(x) - x$. By Intermediate Value Theorem, there exists some $c \in (0, 2023)$ such that $g(c) = 0$. That is $f(c) = c$. We now apply Mean Value Theorem on the intervals $(0, c)$ and $(c, 2023)$ to obtain a, b such that

$$f'(a) = \frac{f(c) - f(0)}{c - 0} = \frac{f(c) - 2023}{c} = \frac{c - 2023}{c}$$

and

$$f'(b) = \frac{f(2023) - f(c)}{2023 - c} = \frac{-c}{2023 - c}.$$

Thus,

$$f'(a)f'(b) = \frac{c - 2023}{c} \frac{-c}{2023 - c} = 1.$$

$\square$

8. Consider the matrices $A = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$. We play a game starting from the matrix A. At every step, call the current matrix C and we either:

   - Swap the row of C that contains the entry 2 with any neighboring (above or below) row of C, or
   - Swap the column of C that contains 2 with any neighboring (to the left or to the right) column of C.

   Prove that we can never reach the matrix B starting from the matrix A, regardless of the number of steps used.

   Solution. Let C_n be the matrix right before step n, so $C_1 = A$. Since the admissible operations are (particular cases of) one row swap or one column swap, we have $\det C_{n+1} = -\det C_n$. Note that $\det A = 2$ and $\det B = -2$. Thus we would need an odd number of swaps to reach $B = C_n$ from $A = C_1$. However we have the extra restriction that we have to bring the entry 2 back to its starting position. Let's imagine how 2 moves along the steps. At every step it moves up, down, to the right, or to the left by one. Since it goes back to its starting point, there must be as many moves to the right as there are to the left, and as many moves down as there are back up. Thus we need an even number of steps. $\square$