Stuart Sidney Math Competition **Name (Print):** ______________________
Spring 2023
04/04/23
Time Limit: 120 Minutes

This exam contains 3 pages (including this cover page) and 8 problems. Check to see if any pages are missing. Write your name on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use books, notes, or any electronic device on this exam.

Problem	Points	Score
1	5	
2	5	
3	5	
4	5	
5	5	
6	5	
7	5	
8	5	
Total:	40	

1. (5 points) Show that $\sin x + \tan x > 2x$ for all $x \in (0, \pi/2)$.

2. (5 points) Let $n \in \mathbb{N}$. Evaluate

$$\int_0^{\pi/2} \frac{\sin^n x}{\cos^n x + \sin^n x} \, \mathrm{d}x.$$

3. (5 points) Let $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $R_\theta(a, b) = (a\cos\theta - b\sin\theta, a\sin\theta + b\cos\theta)$. This is the counterclockwise rotation by θ degrees around the origin $(0, 0)$. Let $M_x(a, b) = (a, -b)$ and $M_y(a, b) = (-a, b)$ be the reflections across the x and y-axes respectively. Let

$$T(a, b) = R_{45}(M_x(R_{90}(M_y(R_{45}(a, b))))).$$

Define $T^2(a, b) = T(T(a, b))$ and similarly define $T^n(a, b)$ for all n. Compute

$$T^{2023}(a, b)$$

in terms of a, b.

4. (5 points) Let $a_n = \sum_{k=1}^{n} \frac{1}{k}$.

    1. Show that the limit
    $$\gamma := \lim_{n\to\infty} (a_n - \log n)$$
    exists in $\mathbb{R}$.
    2. Find the limit
    $$\lim_{n\to\infty} e^{a_{n+1}} - e^{a_n}.$$

5. (5 points) Determine which of the digits $0, 1, 2, 3, 4, 5, 6, 7, 8$ or 9 may occur as the last digit of n^n where n is a positive integer.

6. (5 points) Prove that for all $n \in \mathbb{N}$,

$$\arctan(n + 1) - \arctan(n) < \frac{1}{n^2 + n}.$$

7. (5 points) Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that $f(0) = 2023$ and $f(2023) = 0$. Show that there exist $a, b \in \mathbb{R}$ such that $f'(a)f'(b) = 1$.

8. (5 points) Consider the matrices $A = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$. We play a game starting from the matrix A. At every step, call the current matrix C and we either:

- Swap the row of C that contains the entry 2 with any neighboring (above or below) row of C, or
- Swap the column of C that contains 2 with any neighboring (to the left or to the right) column of C.

Prove that we can never reach the matrix B starting from the matrix A, regardless of the number of steps used.