2019 STUART SIDNEY CALCULUS COMPETITION

Tuesday 26 March, 2018, 6:30–8:00 p.m.

Please show enough of your work to make your line of reasoning clear. Numerical answers will receive no credit if they are not adequately supported. Calculators are welcome, but unlikely to be very useful. Have fun, and good luck!

(1) Let $f(x)$ be a function that is odd and differentiable on $(-\infty, +\infty)$.
  (a) Prove that its derivative $f'(x)$ is an even function.
  (b) Is the converse statement true?

(2) Recall the equality from geometric series:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots, \qquad \text{for } |x| < 1.$$

  (a) Compute the limit of the power series

$$1 \cdot 2 + (2 \cdot 3)x + (3 \cdot 4)x^2 + (4 \cdot 5)x^3 + (5 \cdot 6)x^4 + \cdots \qquad |x| < 1.$$

  as a rational function in x;
  (b) Compute

$$1 - \frac{1 \cdot 2}{2} + \frac{2 \cdot 3}{2^2} - \frac{3 \cdot 4}{2^3} + \frac{4 \cdot 5}{2^4} - \cdots$$

(3) Construct one polynomial $f(x)$ with real coefficients and all of the following properties:
  (a) $f(x)$ is an even function, in other words, $f(x) = f(-x)$,
  (b) $f(2) = f(-2) = 0$,
  (c) $f(x) > 0$ when $-2 < x < 2$, and
  (d) the absolute maximum of $f(x)$ is achieved at $x = 1$ and $x = -1$.
  Justify your answer.

(4) Consider the first quadrant quarter unit disk

$$QD = \left\{(x, y) \;\middle|\; x \geq 0,\ y \geq 0,\ x^2 + y^2 \leq 1\right\}.$$

  Assuming uniform density, find the coordinates of the center of mass of QD. (Hint: the identity $\sin 3x = 3 \sin x - 4 \sin^3 x$ might be helpful.)

(5) Which one of the numbers

$$\int_0^\pi e^{\sin^2 x} dx \quad \text{and} \quad \frac{3\pi}{2}$$

  is larger? Justify your answer.

(6) Let $f : \mathbb{R} \to \mathbb{R}$ be a periodic continuous function, of period $T > 0$, that is $f(x+T) = f(x)$ for any $x \in \mathbb{R}$. Prove that

$$\lim_{x \to \infty} \frac{1}{x} \int_0^x f(t)\, dt = \frac{1}{T} \int_0^T f(t)\, dt.$$

(7) Suppose $(a_n)_{n\geq 1}$ is a <u>decreasing sequence with positive terms</u> such that

$$\sum_{n=1}^{\infty} a_n < \infty.$$

Prove that:

(a) The sequence $x_n = (a_1 + a_2 + \cdots + a_n) - na_n$ is bounded and increasing.

(b) The sequence $(na_n)_{n\geq 1}$ converges to zero.

(8) Find all absolute minimum points for the function $f(x, y) = x^4 + y^4 - 4xy$, where $x, y \in \mathbb{R}$.

(9) Compute

$$\iiint\limits_S \frac{dxdydz}{(1 + x + y + z)^2},$$

where $S = \{x \geq 0, y \geq 0, z \geq 0, x + y + z \leq 1\}$.